TR and ELSV for double Hurwitz numbers
(topological recursion)

Notations $D \geq 1$ A partition of D $(\lambda \vdash D)$ is $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$

$|\lambda| = D$ size

$\ell(\lambda) = \ell$ length.

$\#\mathrm{Aut}(\lambda) = \prod_{m \geq 1} \#\{ i \mid \lambda_i = m\}!$

C_λ conjugacy class in $\mathfrak{S}_D$ (cycle length $\lambda_1, \ldots, \lambda_\ell$)

→ $Z\mathbb{Q}[\mathfrak{S}_D]$ has basis $\hat{C}_\lambda = \sum_{\gamma \in C_\lambda} \gamma$

$\lambda, \mu \vdash D$
$k \geq 0$

$$H^{\bullet, k}_{\lambda,\mu} = \frac{1}{D!} [\mathrm{id}]\, \hat{C}_\lambda \frac{\hat{C}^k_{(2,1,\ldots,1)}}{k!} \hat{C}_\mu$$

$$= \frac{1}{D!}\frac{1}{k!} \# \left\{ \begin{matrix} \sigma_0 \in C_\lambda \\ \sigma_\infty \in C_\mu \end{matrix} \quad \tau_1 \ldots \tau_k \text{ transpositions}, \ \sigma_0 \tau_1 \cdots \tau_k \sigma_\infty = \mathrm{id} \right\}$$

$$= \frac{1}{D!k!} \# \left\{ \rho \in \mathrm{Hom}(\pi_1(S^2 \setminus \{0, \infty, p_1 \ldots p_k\}), \mathfrak{S}_D) \ \middle| \ \begin{matrix} \rho(\gamma_0) \in C_\lambda, \ \rho(\gamma_\infty) \in C_\mu \\ \rho(\gamma_i) \text{ transpositions} \end{matrix} \right\}$$

monodromy of f

$$= \frac{1}{k!} \sum_f{}' \frac{1}{\#\mathrm{Aut}(f)}$$

$f : \Sigma \to \mathbb{P}^1$
branched covers
ramification λ over 0
μ over ∞
k simple branchpoints.

$[H^\circ_{\lambda,\mu}]_g$ by inclusion – exclusion.
count connected cover / transitive $\mathrm{Im}\,\rho$

$2g - 2 + \ell(\lambda) + \ell(\mu) = k$. g = genus.
(Riemann – Hurwitz).

double Hurwitz μ arbitrary
simple $\mu = (1, \ldots, 1)$
d-orbifold. $\mu = (d, \ldots, d)$

- many variants : $\underline{B} \in Z\mathbb{Q}[\mathfrak{S}_D]$ replace $\frac{C^k_{(2,\ldots)}}{k!}$

I SIMPLE HURWITZ NUMBERS.

Ekedahl, Lando, Shapiro, Vainshtein

$\mathcal{H}_{g,\lambda}$ moduli of $(C_g, p_1 \ldots p_n, z_1 \ldots z_k, \pi : C_g \to \mathbb{P}^1)$
with $\mathrm{ord}_{p_i} \pi = \lambda_i$ pole at p_i
$\mathrm{ord}_{z_i} d\pi = 1$

$$\mathcal{L}_{g,\lambda} : \mathcal{H}_{g,\lambda} \longrightarrow \mathbb{C}^k / \mathbb{C}^*$$
$$(C, \pi) \longmapsto [\pi(z)]$$

$$\frac{\deg \mathcal{L}_{g,\lambda}}{k!} = [H^\circ_\lambda]_g$$

ELSV formula (99) express it via intersection theory

$$[H^\circ_\lambda]_g = \underbrace{\prod_{i=1}^n \frac{\lambda_i^{\lambda_i}}{\lambda_i!}}_{\text{special factor.}} \underbrace{\int_{\overline{\mathcal{M}}_{g,n}} \frac{\boxed{\Lambda^\vee}}{\prod_{i=1}^n (1 - \lambda_i \psi_i)}}_{\text{polynomial in } \lambda}$$

dual Hodge class.

$2g - 2 + n > 0$
$n = \ell(\lambda)$

from combinatorics ; deduce

$$1 + \sum_{g \geq 1} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\sum_{h=0}^g \lambda_h (-1)^{g-h} D^{g-h}}{(1 - D\psi_1)} = \left(\frac{t/2}{\mathrm{sh}\, t/2} \right)^{D+1}$$

$\lambda = (D)$

idea

$$\mathcal{H}_{g,\lambda} \overset{\Phi}{\hookrightarrow} \mathcal{P} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

principal parts of π at the poles

$$\mathcal{P}' = \{ (\pi_1, \ldots, \pi_n) \mid \sum_i \mathrm{Res}_{p_i} (\pi_i \omega) = 0 \quad \forall \omega \in H^0(K_C) \}$$

compactify $\overline{\mathcal{P}'} \longrightarrow \overline{\mathcal{M}}_{g,n}$ cone

extract deg by top. Segre class computation

Hodge bundle

- Periods of multidifferentials on analytic curve

→ $$[H^\circ_{(D)}]_{g=0} = \frac{D^{D-2}}{D!}$$ (Cayley tree formula) $g = 0, n = 1$

→ $$[H^\circ_{(\lambda_1 \lambda_2)}]_{g=0} = \frac{\lambda_1^{\lambda_1} \lambda_2^{\lambda_2}}{(\lambda_1 + \lambda_2) \lambda_1! \lambda_2!}$$ (Cayley decorated by trees) $g = 0, n = 2$.

$\mathcal{S} = \{ (X,y) \in \mathbb{C}^* \times \mathbb{C} \mid X = ye^{-y} \}$

$\omega_{0,1} = y \frac{dX}{X}$ $\qquad \operatorname*{Res}_{X=0} \frac{X^{-D}}{D} \omega_{0,1} = [H^0_{(D)}]_{g=0}$

$\omega_{0,2} = \frac{dy_1 dy_2}{(y_1-y_2)^2}$ $\qquad \operatorname*{Res}_{X_1=0} \operatorname*{Res}_{X_2=0} \frac{X_1^{-\lambda_1}}{\lambda_1} \frac{X_2^{-\lambda_2}}{\lambda_2} \omega_{0,2}$
$= [H^0_{(\lambda_1,\lambda_2)}]_{g=0}$

Prediction from mirror symmetry (for $\mathbb{C}^3$)
(Bouchard–Mariño)

$A = \overset{\text{zeroes of}}{dX} = \{(e^{-1}, y=1)\}$.

- qualitative: exist $\omega_{g,n} \in H^0(K_{\mathcal{S}}(*A)^{\boxtimes n}, \mathcal{S}^n)^{\mathfrak{S}_n}$
  such that $[H^0_\lambda]_g = \operatorname*{Res}_{X_i=0} \prod_{i=1}^{n} \frac{X_i^{-\lambda_i}}{\lambda_i} \omega_{g,n}$.
- quantitative $\omega_{g,n}$ satisfy topological recursion $\nearrow$ on $2g-2+n$
  (Eynard–Orantin 07). attached $(\mathcal{S}, X, y, \omega_{0,2})$

proved by Eynard, Mulase, Safnuk 11

In brief

initial data
- $\mathcal{S}$ smooth curve
- x, y meromorphic functions.
  $A = dx^{-1}\{0\}$ finite, disjoint from $dy^{-1}\{0\}$

simplify: zeroes of dx are simple
local Galois involution near $a \in A$ $\qquad x \circ \sigma_a = x$

- $\omega_{0,2} \in H^0(K_{\mathcal{S}}^{\boxtimes 2}(2\Delta), \mathcal{S}^2)^{\mathfrak{S}_2}$ $\qquad \operatorname{res}_\Delta \omega_{0,2} = 1$
  $\Delta =$ diagonal in $\mathcal{S}\times\mathcal{S}$.

$\overset{\text{TR}}{\rightsquigarrow}$ $\omega_{g,n}$ $\qquad g \geq 0,\ n > 0 \qquad \underline{2g-2+n} > 0$.

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{a \in A} \operatorname*{Res}_{z=a} \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1, \cdot)}{\big(y(z)-y(\sigma_a(z))\big)dx(z)} E_{g,n}(z, \sigma_a(z), z_2, \ldots z_n)$$

$$E_{g,n}(z, z', z_2, \ldots, z_n) = \omega_{g-1,n+1}(z, z', z_2, \ldots, z_n) + \overset{\text{no } \omega_{0,1}}{\sum_{\substack{h+h'=g \\ J \cup J' = \{z_2, \ldots, z_n\}}}} \omega_{h,1+|J|}(z, J)\, \omega_{h',1+|J'|}(z', J')$$

- TWO ppT

* analytic variation of initial data $\mathcal{C}_t = (\mathcal{S}_t, x_t, y_t, \omega_{0,2}^t)_{t \in \mathbb{C}}$ $\qquad$ (Eynard–Orantin 07)

if $(\partial_t y\, dx_t - \partial_t x\, dy_t)(z) = \int_{\gamma_t \ni z'} \Upsilon_t(z')\, \omega_{0,2}(z,z')$ and $\partial_t \omega_{0,2}^t(z_1,z_2)|_{x_t(z_i) \text{ fixed}} = \int_{\gamma_t} \Upsilon_t(z')\, \omega_{0,3}(z', z_1, z_2)$

(γ_t, Υ_t) current: γ_t Jordan curve on $\mathcal{S}_t - A$
Υ_t germ of holomorphic function on γ_t

(automatic when $\omega_{0,2} \in H^0(K_{\mathbb{P}^1}^{\boxtimes 2}(2\Delta), (\mathbb{P}^1)^2)$)

then $\partial_t \omega_{g,n}(z_1, \ldots, z_n)|_{x_t(z_i) \text{ fixed}} = \int_{\gamma_t \ni z'} \Upsilon_t(z')\, \omega_{g,n+1}(z', z_1, \ldots, z_n)$ $\qquad$ for $2g-2+n > 0$.

* any initial data $\mathcal{C} = (\mathcal{S}, x, y, \omega_{0,2}) \overset{\text{Eynard '11}}{\rightsquigarrow} \Omega_{\mathcal{C}} \in H^0(\mathbb{C}^A, H^\bullet(\overline{\mathcal{M}}_{g,n}))$

on a suitable basis of $K_{\mathcal{C}}(*A)$ $\qquad (\xi_{k,a})_{\substack{a \in A \\ k \geq 0}}$.

$$\omega_{g,n}(z_1, \ldots, z_n) \overset{\downarrow}{=} \sum_{\substack{k_1, \ldots, k_n \geq 0 \\ a_1, \ldots, a_n \in A}} \left(\int_{\overline{\mathcal{M}}_{g,n}} \Omega_{\mathcal{C}}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \prod_{i=1}^{n} \psi_i^{d_i} \right) \prod_{i=1}^{n} \xi_{k_i, a_i}(z_i)$$

- for $\mathcal{S} = \mathbb{C}$
  $x(z) = z^2/2$
  $y(z) = -z$
  $\underline{\Omega_{\mathcal{C}}}$ fundamental class of $\overline{\mathcal{M}}_{g,n}$
  (Virasoro part of Witten's conj/ Kontsevich theorem).

- deformations :

any semisimple CohFT is represented in this way (but $\Omega_{\mathcal{C}}$ not always a CohFT in general)

deformation $\simeq$ Givental gp action.

- $\underline{\mathcal{S}}$: $X = ye^{-y}$ $\qquad \Omega_{\mathcal{C}} = \Lambda^\vee$
  $x = \ln X$

$\omega_{g,n}$ defined on $\mathcal{S}$ $\longleftrightarrow$ quasi-polynomiality

ELSV $\Longleftrightarrow$ TR for $\mathcal{S}$: $X = ye^{-y}$.

II Double Hurwitz numbers

more fundamental $[H^{\circ}_{\lambda,\mu}]_g$

- integrability manifest
- double ramification theory ⇝ piecewise polynomiality
  wall crossing

not obvious to obtain ELSV-like formula

Goulden–Jackson conj + Cavalieri

- Enough to truncate $\max_i \mu_i \leq d$. We will give to λ and μ a dissymmetric role
- Johnson – Pandharipande – Tseng. (11).

$\overline{\mathcal{M}}_{g,a_1,\dots,a_n}(\mathbb{P}^1[d],\lambda)$ $a_1,\dots,a_n \in \{0,\dots,d-1\}$

$d \mid \underset{|\lambda|}{\overset{\shortparallel}{D}} - \Sigma a_i$

$\mathbb{P}^1[d]$: $\mathbb{P}^1$ stacky point order d at ∞
λ : relative condition at 0
$a_1,\dots,a_n$ monodromies.

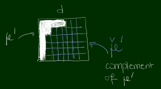
$$\begin{tikzcd} \mathbb{E}^U \arrow[r, phantom, "\subset"] & \mathbb{E} \arrow[r] & \overline{\mathcal{M}}_{g,a_1\dots a_n}(B\mathbb{Z}_d) \arrow[d, "\varepsilon"] \\ & & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

$\mathbb{E}^U$: on which generator acts by $e^{2i\pi/d}$

$\mathbb{E}$: Hodge bundle

$\Lambda^{[d]}_{g,a_1,\dots,a_n} \in \varepsilon_*\left[\sum (-1)^h c_h(\mathbb{E}^U)\right]$

- $\max_i \mu_i \leq d$ $\underline{\mu'}$ = μ minus the parts of size d.



Def μ' satisfy nonnegativity if $D - |\mu'| \geq 0$

boundedness if $\max_{i \neq j} (\check{\mu}'_i + \check{\mu}'_j) \geq d$

When μ' non negative ⇝ $\mu = \mu' \cup (d,\dots,d)$ so that $|\mu| = |\lambda| = D$.
If also boundedness, then degree of branching map gives $[H^{\circ}_{\lambda,\mu}]_g$ and we have

$$[H^{\circ}_{\lambda,\mu}]_g = d^{2g-2+n+\ell(\lambda)} \prod_{i=1}^{n} \frac{(\lambda_i/d)^{\lfloor \lambda_i/d \rfloor}}{\lfloor \lambda_i/d \rfloor !} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda^{[d]}_{g,-\bar{\lambda},\mu}}{\prod_{i=1}^{n}\left(1 - \frac{\lambda_i}{d}\psi_i\right)}$$

JPT formula

where $-\bar{\lambda}_i$ is the unique element of $\{0,\dots,d-1\}$ such that $-\bar{\lambda} = -\lambda_i \bmod d$.

If negativity and boundedness, the right-hand side vanish.

JPT vanishing

Boundedness implies that boundary strata do not give further contributions to the degree of branching map

Thm (B, Do, Karev, Lewanski, Moskovsky 20)

① Let $d \geq 1$, $q_1,\dots,q_d \in \mathbb{C}$, $q_d \neq 0$, $Q(z) = \sum_{j=1}^{d} q_j z^j$

TR for the initial data
$\mathcal{S} = \mathbb{C}$, $X = z e^{-Q(z)}$, $x = \ln X$
$y = Q(z)$
$\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}$

⇝ $(\omega_{g,n})_{2g-2+n>0}$

satisfies $$\operatorname*{Res}_{X_1=0} \cdots \operatorname*{Res}_{X_n=0} \left[\prod_{i=1}^{n} \frac{X_i^{-\lambda_i}}{\lambda_i}\, \omega_{g,n}\right] = \sum_{\mu \vdash |\lambda|} \frac{\prod_{i=1}^{\ell(\mu)} q_{\mu_i}}{\# \mathrm{Aut}(\mu)} [H^{\circ}_{\lambda,\mu}]_g$$

② For any $\lambda, \mu \vdash D$, we have an ELSV-like formula

$$[H^\circ_{\lambda,\mu}]_g = d^{2g-2+n+\ell(\mu)} \prod_{i=1}^{n} \frac{(\lambda_i/d)^{\lfloor \mu_i/d \rfloor}}{\lfloor \mu_i/d \rfloor!}$$

$\ell(\lambda)=n$

$2g-2+n>0$

$$\times \left\{ \sum_{m=0}^{\ell(\mu')} \frac{(-1)^{\ell(\mu')-m}}{m!} \sum_{\substack{\rho^1\sqcup\cdots\sqcup\rho^m=\check{\mu}' \\ \max_j \rho^i_j \leq d-1}} \frac{\prod_{i=1}^{m} \left[\frac{d-|\rho^i|}{d}\right]_{\ell(\rho^i)-1}}{\#\mathrm{Aut}\,\rho^i} \int_{\overline{\mathcal{M}}_{g,n+m}} \frac{\Lambda^{[d]}_{-\bar\lambda,\, d-|\rho^1|,\dots,d-|\rho^m|}}{\prod_{i=1}^{n}\left(1-\frac{\lambda_i}{d}\psi_i\right)} \right\}$$

where $[x]_\ell = x(x+1)\cdots(x+\ell-1)$ Pochhammer symbol.

If $\mu'=\emptyset$, $m=0$ only term

If $\mu'\neq\emptyset$, $m\in\{1,\dots,\ell(\mu')\}$ only

③ If $|\lambda|-|\mu|<0$ the right-hand side vanish.

This sum always contain $m=\ell(\mu')$ $\rho^i=(\mu'_i)$ $i=1,\dots,\ell(\mu')$

that is the JPT term $\propto \int_{\overline{\mathcal{M}}_{g,n+m}} \frac{\Lambda^{[d]}_{-\lambda,\mu}}{\prod_{i=1}^{n}\left(1-\frac{\lambda_i}{d}\psi_i\right)}$

There are other terms iff boundedness is violated

$\rightsquigarrow$ this is an inconditional generalisation of JPT

Same approach may be applicable to other type of Hurwitz numbers (double r-spin, mixed completed cycles in view of GW/H correspondence)

Strategy of proof

① prove quasipolynomiality

starting point : Okounkov - Pandharipande formula for

$[H^{\circ,k}_{\lambda,\mu}]$ as $\langle C_\mu(\lambda_1)\cdots C_\mu(\lambda_n)\rangle$ in semi-infinite wedge $\Lambda^{\infty}_{2}\mathbb{C}((z))$

commute operators and extract (hard part)

the quasipolynomiality

$\Rightarrow$ implies generating series $\sum_{\substack{|\lambda|=|\mu| \\ \ell(\lambda)=n}} [H^\circ_{\lambda,\mu}]_g \prod_{i=1}^{n} d(X_i^{\lambda_i}) \frac{\prod_{j=1}^{\ell(\mu)} q_{\mu_j}}{\#\mathrm{Aut}\,\mu} = \tilde\omega_{g,n}$

analytically continues to $\mathcal{S}$ given in the theorem with poles at $A=\{dx=0\}$

② cut-and-join equations give recursion on k for $H^{\circ,k}_{\mu}$

(easy from semi-infinite wedge formalism)

$\rightsquigarrow$ translate into functional relations for generating series

look at their analytic continuation on $\mathcal{S}$

this gives recursion on polar part of $\tilde\omega_{g,n}$ at A

use residue formula to compute $\tilde{\omega}_{g,n} \longrightarrow$ TR formula

③ we already know the ELSV-like formula for the d-orbifold case (JPT for $\mu' = \emptyset$) and TR [Do–Leigh–Norbury 12
Bouchard, Hernandez-Serrano, Liu, Mulase 12]

They are in fact equivalent (Ω_ϵ is the $\Lambda^{[d]}$ class in that case)

[Lewanski, Popolitov, Shadrin, Zvonkine]

use the family of curves

$$\left(\Sigma_t = \mathbb{C} \quad \begin{aligned} \cancel{X}_t(z) &= z e^{-Q_t(z)} \\ y_t(z) &= Q_t(z) \\ Q_t(z) &= q_d z^d + t \sum_{j=1}^{d-1} q_j z^j \end{aligned} \right) = \mathcal{C}_t$$

to interpolate between

$\omega_{g,n}^{t=0} \rightsquigarrow$ d-orbifold

$\omega_{g,n}^{t=1} \leftrightarrow$ double

and write $$\omega_{g,n}^{t=1}(z_1,\dots,z_n) = \sum_{k \geq 0} \frac{\partial_t^k \omega_{g,n}}{k!}(z_1,\dots,z_n) \Big|_{x_t(z_i) \text{ fixed}} \Big|_{t=0}$$

For this specific $\mathcal{C}_t$, we can use the variation formula I mentioned but beware we work at $x_t(z_i)$ fixed and (Υ_t is naturally expressed as a function of z)

$\rightsquigarrow$ use Faà di Bruno formula to compute the n-th order derivative

creates the complicated sum over subpartitions of μ'

Demands some computations but no guess

It would be interesting to see the same structure arising by inclusion-exclusion to get rid of contributions of boundary strata in the JPT approach on $\overline{\mathcal{M}}_{g,a_1,\dots,a_n}(\mathbb{P}^1[d],\lambda)$